

Homotopy types of strict 3-groupoids

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It has been difficult to see precisely the role played by *strict* n -categories in the nascent theory of n -categories, particularly as related to n -truncated homotopy types of spaces. We propose to show in a fairly general setting that one cannot obtain all 3-types by any reasonable realization functor¹ from strict 3-groupoids (i.e. groupoids in the sense of [20]). More precisely we show that one does not obtain the 3-type of S^2 . The basic reason is that the Whitehead bracket is nonzero. This phenomenon is actually well-known, but in order to take into account the possibility of an arbitrary reasonable realization functor we have to write the argument in a particular way.

We start by recalling the notion of strict n -category. Then we look at the notion of strict n -groupoid as defined by Kapranov and Voevodsky [20]. We show that their definition is equivalent to a couple of other natural-looking definitions (one of these equivalences was left as an exercise in [20]). At the end of these first sections, we have a picture of strict 3-groupoids having only one object and one 1-morphism, as being equivalent to abelian monoidal objects $(G, +)$ in the category of groupoids, such that $(\pi_0(G), +)$ is a group. In the case in question, this group will be $\pi_2(S^2) = \mathbf{Z}$. Then comes the main part of the argument. We show that, up to inverting a few equivalences, such an object has a morphism giving a splitting of the Postnikov tower (Proposition 5.3. It follows that for any realization functor respecting homotopy groups, the Postnikov tower of the realization (which has two stages corresponding to π_2 and π_3) splits. This implies that the 3-type of S^2 cannot occur as a realization.

The fact that strict n -groupoids are not appropriate for modelling all homotopy types has in principle been known for some time. There are several papers by R. Brown and coauthors on this subject, see [9], [10], [11], [12]; a recent paper by C. Berger [8]; and also a discussion of this in various places in Grothendieck [18]. Other related examples are given in Gordon-Power-Street [17]. The novelty of our present treatment is that we have written the argument in such a way that it applies to a wide class of possible realization functors, and in particular it applies to the realization functor of Kapranov-Voevodsky (1991) [20].

¹ Our notion of “reasonable realization functor” (Definition 3.1) is any functor \mathfrak{R} from the category of strict n -groupoids to Top , provided with a natural transformation r from the set of objects of G to the points of $\mathfrak{R}(G)$, and natural isomorphisms $\pi_0(G) \cong \pi_0(\mathfrak{R}(G))$ and $\pi_i(G, x) \cong \pi_i(\mathfrak{R}(G), r(x))$. This axiom is fundamental to the question of whether one can realize homotopy types by strict n -groupoids, because one wants to read off the homotopy groups of the space from the strict n -groupoid. The standard realization functors satisfy this property, and the somewhat different realization construction of [20] is claimed there to have this property.

This problem with strict n -groupoids can be summed up by saying in R. Brown's terminology, that they correspond to *crossed complexes*. While a nontrivial action of π_1 on the π_i can occur in a crossed complex, the higher Whitehead operations such as $\pi_2 \otimes \pi_2 \rightarrow \pi_3$ must vanish. This in turn is due to the fundamental “interchange rule” (or “Godement relation” or “Eckmann-Hilton argument”). This effect occurs when one takes two 2-morphisms a and b both with source and target a 1-identity 1_x . There are various ways of composing a and b in this situation, and comparison of these compositions leads to the conclusion that all of the compositions are commutative. In a weak n -category, this commutativity would only hold up to higher homotopy, which leads to the notion of “braiding”; and in fact it is exactly the braiding which leads to the Whitehead operation. However, in a strict n -category, the commutativity is exact, so the Whitehead operation is trivial.

One can observe that one of the reasons why this problem occurs is that we have the exact 1-identity 1_x . This leads to wondering if one could get a better theory by getting rid of the exact identities. We speculate in this direction at the end of the paper by proposing a notion of *n -snucategory*, which would be an n -category with strictly associative composition, but without units; we would only require existence of weak units. The details of the notion of weak unit are not worked out.

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1. Strict n -categories

In what follows *all n -categories are meant to be strict n -categories*. For this reason we try to put in the adjective “strict” as much as possible when $n > 1$; but in any case, the very few times that we speak of weak n -categories, this will be explicitly stated. We mostly restrict our attention to $n \leq 3$.

In case that isn't already clear, it should be stressed that everything we do in this section (as well as most of the next and even the subsequent one as well) is very well known and classical, so much so that I don't know what are the original references.

To start with, a *strict 2-category* A is a collection of objects A_0 plus, for each pair of objects $x, y \in A_0$ a category $Hom_A(x, y)$ together with a morphism

$$Hom_A(x, y) \times Hom_A(y, z) \rightarrow Hom_A(x, z)$$

which is strictly associative in the obvious way; and such that a unit exists, that is an element $1_x \in Ob Hom_A(x, x)$ with the property that multiplication by 1_x acts trivially

on objects of $Hom_A(x, y)$ or $Hom_A(y, x)$ and multiplication by 1_{1_x} acts trivially on morphisms of these categories.

A *strict 3-category* C is the same as above but where $Hom_C(x, y)$ are supposed to be strict 2-categories. There is an obvious notion of direct product of strict 2-categories, so the above definition applies *mutatis mutandis*.

For general n , the well-known definition is most easily presented by induction on n . We assume known the definition of strict $n - 1$ -category for $n - 1$, and we assume known that the category of strict $n - 1$ -categories is closed under direct product. A *strict n -category* C is then a category enriched [21] over the category of strict $n - 1$ -categories. This means that C is composed of a *set of objects* $Ob(C)$ together with, for each pair $x, y \in Ob(C)$, a *morphism-object* $Hom_C(x, y)$ which is a strict $n - 1$ -category; together with a strictly associative composition law

$$Hom_C(x, y) \times Hom_C(y, z) \rightarrow Hom_C(x, z)$$

and a morphism $1_x : * \rightarrow Hom_C(x, x)$ (where $*$ denotes the final object cf below) acting as the identity for the composition law. The *category of strict n -categories* denoted $nStrCat$ is the category whose objects are as above and whose morphisms are the transformations strictly perserving all of the structures. Note that $nStrCat$ admits a direct product: if C and C' are two strict n -categories then $C \times C'$ is the strict n -category with

$$Ob(C \times C') := Ob(C) \times Ob(C')$$

and for $(x, x'), (y, y') \in Ob(C \times C')$,

$$Hom_{C \times C'}((x, x'), (y, y')) := Hom_C(x, y) \times Hom_{C'}(x', y')$$

where the direct product on the right is that of $(n - 1)StrCat$. Note that the final object of $nStrCat$ is the strict n -category $*$ with exactly one object x and with $Hom_*(x, x) = *$ being the final object of $(n - 1)StrCat$.

The induction inherent in this definition may be worked out explicitly to give the definition as it is presented in [20] for example. In doing this one finds that underlying a strict n -category C are the sets $Mor^i(C)$ of *i -morphisms* or *i -arrows*, for $0 \leq i \leq n$. The 0-morphisms are by definition the objects, and $Mor^i(C)$ is the disjoint union over all pairs x, y of the $Mor^{i-1}(Hom_C(x, y))$. The composition laws at each stage lead to various compositions for i -morphisms, denoted in [20] by $*_j$ for $0 \leq j < i$. These are partially defined depending on the *source* and *target* maps. For a more detailed explanation, refer to the standard references [11] [24] [20] (and I am probably missing many older references which could date back even before [7] [16]).

One of the most important of the axioms satisfied by the various compositions in a strict n -category is variously known under the name of “Eckmann-Hilton argument”,

“Godement relations”, “interchange rules” etc. The following discussion of this axiom owes a lot to discussions I had with Z. Tamsamani during his thesis work. This axiom comes from the fact that the composition law

$$Hom_C(x, y) \times Hom_C(y, z) \rightarrow Hom_C(x, z)$$

is a morphism with domain the direct product of the two morphism $n - 1$ -categories from x to y and from y to z . In a direct product, compositions in the two factors by definition are independent (commute). Thus, for 1-morphisms in $Hom_C(x, y) \times Hom_C(y, z)$ (where the composition $*_0$ for these $n - 1$ -categories is actually the composition $*_1$ for C and we adopt the latter notation), we have

$$(a, b) *_1 (c, d) = (a *_1 c, b *_1 d).$$

This leads to the formula

$$(a *_0 b) *_1 (c *_0 d) = (a *_1 c) *_0 (b *_1 d).$$

This seemingly innocuous formula takes on a special meaning when we start inserting identity maps. Suppose $x = y = z$ and let 1_x be the identity of x which may be thought of as an object of $Hom_C(x, x)$. Let e denote the 2-morphism of C , identity of 1_x ; which may be thought of as a 1-morphism of $Hom_C(x, x)$. It acts as the identity for both compositions $*_0$ and $*_1$ (the reader may check that this follows from the part of the axioms for an n -category saying that the morphism $1_x : * \rightarrow Hom_C(x, x)$ is an identity for the composition).

If a, b are also endomorphisms of 1_x , then the above rule specializes to:

$$a *_1 b = (a *_0 e) *_1 (e *_0 b) = (a *_1 e) *_0 (e *_1 b) = a *_0 b.$$

Thus in this case the compositions $*_0$ and $*_1$ are the same. A different ordering gives the formula

$$a *_1 b = (e *_0 a) *_1 (b *_0 e) = (e *_1 b) *_0 (a *_1 e) = b *_0 a.$$

Therefore we have

$$a *_1 b = b *_1 a = a *_0 b = b *_0 a.$$

This argument says, then, that $Ob(Hom_{Hom_C(x, x)}(1_x, 1_x))$ is a commutative monoid and the two natural multiplications are the same.

The same argument extends to the whole monoid structure on the $n - 2$ -category $Hom_{Hom_C(x, x)}(1_x, 1_x)$:

Lemma 1.1 *The two composition laws on the strict $n - 2$ -category $Hom_{Hom_C(x, x)}(1_x, 1_x)$ are equal, and this law is commutative. In other words, $Hom_{Hom_C(x, x)}(1_x, 1_x)$ is an abelian monoid-object in the category $(n - 2)StrCat$.*

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There is a partial converse to the above observation: if the only object is x and the only 1-morphism is 1_x then nothing else can happen and we get the following equivalence of categories.

Lemma 1.2 *Suppose G is an abelian monoid-object in the category $(n-2)StrCat$. Then there is a unique strict n -category C such that*

$$Ob(C) = \{x\} \quad \text{and} \quad Mor^1(C) = Ob(Hom_C(x, x)) = \{1_x\}$$

and such that $Hom_{Hom_C(x, x)}(1_x, 1_x) = G$ as an abelian monoid-object. This construction establishes an equivalence between the categories of abelian monoid-objects in $(n-2)StrCat$, and the strict n -categories having only one object and one 1-morphism.

Proof: Define the strict $n-1$ -category U with $Ob(U) = \{u\}$ and $Hom_U(u, u) = G$ with its monoid structure as composition law. The fact that the composition law is commutative allows it to be used to define an associative and commutative multiplication

$$U \times U \rightarrow U.$$

Now let C be the strict n -category with $Ob(C) = \{x\}$ and $Hom_C(x, x) = U$ with the above multiplication. It is clear that this construction is inverse to the previous one. ///

It is clear from the construction (the fact that the multiplication on U is again commutative) that the construction can be iterated any number of times. We obtain the following corollary.

Corollary 1.3 *Suppose C is a strict n -category with only one object and only one 1-morphism. Then there exists a strict $n+1$ -category B with only one object b and with $Hom_B(b, b) \cong C$.*

Proof: By the previous lemmas, C corresponds to an abelian monoid-object G in $(n-2)StrCat$. Construct U as in the proof of 1.2, and note that U is an abelian monoid-object in $(n-1)StrCat$. Now apply the result of 1.2 directly to U to obtain $B \in (n+1)StrCat$, which will have the desired property. ///

2. The groupoid condition

Recall that a *groupoid* is a category where all morphisms are invertible. This definition generalizes to strict n -categories in the following way [20]. We give a theorem stating that three versions of this definition are equivalent.

Note that, following [20], we *do not* require strict invertibility of morphisms, thus the notion of strict n -groupoid is more general than the notion employed by Brown and Higgins [11].

Our discussion is in many ways parallel to the treatment of the groupoid condition for weak n -categories in [25] and our treatment in this section comes in large part from discussions with Z. Tamsamani about this.

The statement of the theorem-definition is recursive on n .

Theorem 2.1 *Fix $n < \infty$.*

I. Groupoids *Suppose A is a strict n -category. The following three conditions are equivalent (and in this case we say that A is a strict n -groupoid).*

- (1) *A is an n -groupoid in the sense of Kapranov-Voevodsky [20];*
- (2) *for all $x, y \in A$, $\text{Hom}_A(x, y)$ is a strict $n - 1$ -groupoid, and for any 1-morphism $f : x \rightarrow y$ in A , the two morphisms of composition with f*

$$\text{Hom}_A(y, z) \rightarrow \text{Hom}_A(x, z), \quad \text{Hom}_A(w, x) \rightarrow \text{Hom}_A(w, y)$$

are equivalences of strict $n - 1$ -groupoids (see below);

- (3) *for all $x, y \in A$, $\text{Hom}_A(x, y)$ is a strict $n - 1$ -groupoid, and $\tau_{\leq 1} A$ (defined below) is a 1-groupoid.*

II. Truncation *If A is a strict n -groupoid, then define $\tau_{\leq k} A$ to be the strict k -category whose i -morphisms are those of A for $i < k$ and whose k -morphisms are the equivalence classes of k -morphisms of A under the equivalence relation that two are equivalent if there is a $k + 1$ -morphism joining them. The fact that this is an equivalence relation is a statement about $n - k$ -groupoids. The set $\tau_{\leq 0} A$ will also be denoted $\pi_0 A$. The truncation is again a k -groupoid, and for n -groupoids A the truncation coincides with the operation defined in [20].*

III. Equivalence *A morphism $f : A \rightarrow B$ of strict n -groupoids is said to be an equivalence if the following equivalent conditions are satisfied:*

- (a) *(this is the definition in [20]) f induces an isomorphism $\pi_0 A \rightarrow \pi_0 B$, and for every object $a \in A$ f induces an isomorphism $\pi_i(A, a) \xrightarrow{\cong} \pi_i(B, f(a))$ where these homotopy groups are as defined in [20];*
- (b) *f induces a surjection $\pi_0 A \rightarrow \pi_0 B$ and for every pair of objects $x, y \in A$ f induces an equivalence of $n - 1$ -groupoids $\text{Hom}_A(x, y) \rightarrow \text{Hom}_B(f(x), f(y))$;*
- (c) *if u, v are i -morphisms in A sharing the same source and target, and if r is an $i + 1$ -morphism in B going from $f(u)$ to $f(v)$ then there exists an $i + 1$ -morphism t in A going from u to v and an $i + 2$ -morphism in B going from $f(t)$ to r (this includes the limiting cases $i = -1$ where u and v are not specified, and $i = n - 1, n$ where “ $n + 1$ -morphisms” mean equalities between n -morphisms and “ $n + 2$ -morphisms” are not specified).*

IV. Sub-lemma *If $f : A \rightarrow B$ and $g : B \rightarrow C$ are morphisms of strict n -groupoids and if any two of f , g and gf are equivalences, then so is the third.*

V. Second sub-lemma *If*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

are morphisms of strict n -groupoids and if hg and gf are equivalences, then g is an equivalence.

Proof: It is clear for $n = 0$, so we assume $n \geq 1$ and proceed by induction on n : we assume that the theorem is true (and all definitions are known) for strict $n - 1$ -categories.

We first discuss the existence of truncation (part II), for $k \geq 1$. Note that in this case $\tau_{\leq k}A$ may be defined as the strict k -category with the same objects as A and with

$$\text{Hom}_{\tau_{\leq k}A}(x, y) := \tau_{\leq k-1}\text{Hom}_A(x, y).$$

Thus the fact that the relation in question is an equivalence relation, is a statement about $n - 1$ -categories and known by induction. Note that the truncation operation clearly preserves any one of the three groupoid conditions (1), (2), (3). Thus we may affirm in a strong sense that $\tau_{\leq k}(A)$ is a k -groupoid without knowing the equivalence of the conditions (1)-(3).

Note also that the truncation operation for n -groupoids is the same as that defined in [20] (they define truncation for general strict n -categories but for n -categories which are not groupoids, their definition is different from that of [25] and not all that useful).

For $0 \leq k \leq k' \leq n$ we have

$$\tau_{\leq k}(\tau_{\leq k'}(A)) = \tau_{\leq k}(A).$$

To see this note that the equivalence relation used to define the k -arrows of $\tau_{\leq k}(A)$ is the same if taken in A or in $\tau_{\leq k+1}(A)$ —the existence of a $k + 1$ -arrow going between two k -arrows is equivalent to the existence of an equivalence class of $k + 1$ -arrows going between the two k -arrows.

Finally using the above remark we obtain the existence of the truncation $\tau_{\leq 0}(A)$: the relation is the same as for the truncation $\tau_{\leq 0}(\tau_{\leq 1}(A))$, and $\tau_{\leq 1}(A)$ is a strict 1-groupoid in the usual sense so the arrows are invertible, which shows that the relation used to define the 0-arrows (i.e. objects) in $\tau_{\leq 0}(A)$ is in fact an equivalence relation.

We complete our discussion of truncation by noting that there is a natural morphism of strict n -categories $A \rightarrow \tau_{\leq k}(A)$, where the right hand side (*a priori* a strict k -category) is considered as a strict n -category in the obvious way.

We turn next to the notion of equivalence (part III), and prove that conditions (a) and (b) are equivalent. This notion for n -groupoids will not enter into the subsequent

treatment of part (I)—what does enter is the notion of equivalence for $n - 1$ -groupoids, which is known by induction—so we may assume the equivalence of definitions (1)-(3) for our discussion of part (III).

Recall first of all the definition of the homotopy groups. Let 1_a^i denote the i -fold iterated identity of an object a ; it is an i -morphism, the identity of 1_a^{i-1} (starting with $1_a^0 = a$). Then

$$\pi_i(A, a) := \text{Hom}_{\tau_{\leq i}(A)}(1_a^{i-1}, 1_a^{i-1}).$$

This definition is completed by setting $\pi_0(A) := \tau_{\leq 0}(A)$. These definitions are the same as in [20]. Note directly from the definition that for $i \leq k$ the truncation morphism induces isomorphisms

$$\pi_i(A, a) \xrightarrow{\cong} \pi_i(\tau_{\leq k}(A), a).$$

Also for $i \geq 1$ we have

$$\pi_i(A, a) = \pi_{i-1}(\text{Hom}_A(a, a), 1_a).$$

One shows that the π_i are abelian for $i \geq 2$. This is part of a more general principle, the “interchange rule” or “Godement relations” referred to in §1.

Suppose $f : A \rightarrow B$ is a morphism of strict n -groupoids satisfying condition (b). From the immediately preceding formula and the inductive statement for $n - 1$ -groupoids, we get that f induces isomorphisms on the π_i for $i \geq 1$. On the other hand, the truncation $\tau_{\leq 1}(f)$ satisfies condition (b) for a morphism of 1-groupoids, and this is readily seen to imply that $\pi_0(f)$ is an isomorphism. Thus f satisfies condition (a).

Suppose on the other hand that $f : A \rightarrow B$ is a morphism of strict n -groupoids satisfying condition (a). Then of course $\pi_0(f)$ is surjective. Consider two objects $x, y \in A$ and look at the induced morphism

$$f^{x,y} : \text{Hom}_A(x, y) \rightarrow \text{Hom}_B(f(x), f(y)).$$

We claim that $f^{x,y}$ satisfies condition (a) for a morphism of $n - 1$ -groupoids. For this, consider a 1-morphism from x to y , i.e. an object $r \in \text{Hom}_A(x, y)$. By version (2) of the groupoid condition for A , multiplication by r induces an equivalence of $n - 1$ -groupoids

$$m(r) : \text{Hom}_A(x, x) \rightarrow \text{Hom}_A(x, y),$$

and furthermore $m(r)(1_x) = r$. The same is true in B : multiplication by $f(r)$ induces an equivalence

$$m(f(r)) : \text{Hom}_B(f(x), f(x)) \rightarrow \text{Hom}_B(f(x), f(y)).$$

The fact that f is a morphism implies that these fit into a commutative square

$$\begin{array}{ccc} \text{Hom}_A(x, x) & \rightarrow & \text{Hom}_A(x, y) \\ \downarrow & & \downarrow \\ \text{Hom}_B(f(x), f(x)) & \rightarrow & \text{Hom}_B(f(x), f(y)). \end{array}$$

The equivalence condition (a) for f implies that the left vertical morphism induces isomorphisms

$$\pi_i(\text{Hom}_A(x, x), 1_x) \xrightarrow{\cong} \pi_i(\text{Hom}_B(f(x), f(x)), 1_{f(x)}).$$

Therefore the right vertical morphism (i.e. $f_{x,y}$) induces isomorphisms

$$\pi_i(\text{Hom}_A(x, y), r) \xrightarrow{\cong} \pi_i(\text{Hom}_B(f(x), f(y)), f(r)),$$

this for all $i \geq 1$. We have now verified these isomorphisms for any base-object r . A similar argument implies that $f^{x,y}$ induces an injection on π_0 . On the other hand, the fact that f induces an isomorphism on π_0 implies that $f^{x,y}$ induces a surjection on π_0 (note that these last two statements are reduced to statements about 1-groupoids by applying $\tau_{\leq 1}$ so we don't give further details). All of these statements taken together imply that $f^{x,y}$ satisfies condition (a), and by the inductive statement of the theorem for $n-1$ -groupoids this implies that $f^{x,y}$ is an equivalence. Thus f satisfies condition (b).

We now remark that condition (b) is equivalent to condition (c) for a morphism $f : A \rightarrow B$. Indeed, the part of condition (c) for $i = -1$ is, by the definition of π_0 , identical to the condition that f induces a surjection $\pi_0(A) \rightarrow \pi_0(B)$. And the remaining conditions for $i = 0, \dots, n+1$ are identical to the conditions of (c) corresponding to $j = i - 1 = -1, \dots, (n-1) + 1$ for all the morphisms of $n-1$ -groupoids $\text{Hom}_A(x, y) \rightarrow \text{Hom}_B(f(x), f(y))$. (In terms of u and v appearing in the condition in question, take x to be the source of the source of the source \dots , and take y to be the target of the target of the target \dots). Thus by induction on n (i.e. by the equivalence $(b) \Leftrightarrow (c)$ for $n-1$ -groupoids), the conditions (c) for f for $i = 0, \dots, n+1$, are equivalent to the conditions that $\text{Hom}_A(x, y) \rightarrow \text{Hom}_B(f(x), f(y))$ be equivalences of $n-1$ -groupoids. Thus condition (c) for f is equivalent to condition (b) for f , which completes the proof of part (III) of the theorem.

We now proceed with the proof of part (I) of Theorem 2.1. Note first of all that the implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are easy. We give a short discussion of $(1) \Rightarrow (3)$ anyway, and then we prove $(3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$.

Note also that the equivalence $(1) \Leftrightarrow (2)$ is the content of Proposition 1.6 of [20]; we give a proof here because the proof of Proposition 1.6 was “left to the reader” in [20].

(1) \Rightarrow (3): Suppose A is a strict n -category satisfying condition (1). This condition (from [20]) is compatible with truncation, so $\tau_{\leq 1}(A)$ satisfies condition (1) for 1-categories; which in turn is equivalent to the standard condition of being a 1-groupoid, so we get that $\tau_{\leq 1}(A)$ is a 1-groupoid. On the other hand, the conditions (1) from [20] for i -arrows, $1 \leq i \leq n$, include the same conditions for the $i-1$ -arrows of $\text{Hom}_A(x, y)$ for any $x, y \in \text{Ob}(A)$ (the reader has to verify this by looking at the definition in [20]). Thus by the inductive

statement of the present theorem for strict $n - 1$ -categories, $Hom_A(x, y)$ is a strict $n - 1$ -groupoid. This shows that A satisfies condition (3).

(3) \Rightarrow (2): Suppose A is a strict n -category satisfying condition (3). It already satisfies the first part of condition (2), by hypothesis. Thus we have to show the second part, for example that for $f : x \rightarrow y$ in $Ob(Hom_A(x, y))$, composition with f induces an equivalence

$$Hom_A(y, z) \rightarrow Hom_A(x, z)$$

(the other part is dual and has the same proof which we won't repeat here).

In order to prove this, we need to make a digression about the effect of composition with 2-morphisms. Suppose $f, g \in Ob(Hom_A(x, y))$ and suppose that u is a 2-morphism from f to g —this last supposition may be rewritten

$$u \in Ob(Hom_{Hom_A(x, y)}(f, g)).$$

Claim: Suppose z is another object; we claim that if composition with f induces an equivalence $Hom_A(y, z) \rightarrow Hom_A(x, z)$, then composition with g also induces an equivalence $Hom_A(y, z) \rightarrow Hom_A(x, z)$.

To prove the claim, suppose that h, k are two 1-morphisms from y to z . We now obtain a diagram

$$\begin{array}{ccc} Hom_{Hom_A(y, z)}(h, k) & \rightarrow & Hom_{Hom_A(x, z)}(hf, kf) \\ \downarrow & & \downarrow \\ Hom_{Hom_A(x, z)}(hg, kg) & \rightarrow & Hom_{Hom_A(x, z)}(hf, kg), \end{array}$$

where the top arrow is given by composition $*_0$ with 1_f ; the left arrow by composition $*_0$ with 1_g ; the bottom arrow by composition $*_1$ with the 2-morphism $h *_0 u$; and the right morphism is given by composition with $k *_0 u$. This diagram commutes (that is the “Godement rule” or “interchange rule” cf [20] p. 32). By the inductive statement of the present theorem (version (2) of the groupoid condition) for the $n - 1$ -groupoid $Hom_A(x, z)$, the morphisms on the bottom and on the right in the above diagram are equivalences. The hypothesis in the claim that f is an equivalence means that the morphism along the top of the diagram is an equivalence; thus by the sub-lemma (part (IV) of the present theorem) applied to the $n - 2$ -groupoids in the diagram, we get that the morphism on the left of the diagram is an equivalence. This provides the second half of the criterion (b) of part (III) for showing that the morphism of composition with g , $Hom_A(y, z) \rightarrow Hom_A(x, z)$, is an equivalence of $n - 1$ -groupoids.

To finish the proof of the claim, we now verify the first half of criterion (b) for the morphism of composition with g (in this part we use directly the condition (3) for A and don't use either f or u). Note that $\tau_{\leq 1}(A)$ is a 1-groupoid, by the condition (3) which we are assuming. Note also that (by definition)

$$\pi_0 Hom_A(y, z) = Hom_{\tau_{\leq 1} A}(y, z) \quad \text{and} \quad \pi_0 Hom_A(x, z) = Hom_{\tau_{\leq 1} A}(x, z),$$

and the morphism in question here is just the morphism of composition by the image of g in $\tau_{\leq 1}(A)$. Invertibility of this morphism in $\tau_{\leq 1}(A)$ implies that the composition morphism

$$Hom_{\tau_{\leq 1}A}(y, z) \rightarrow Hom_{\tau_{\leq 1}A}(x, z)$$

is an isomorphism. This completes verification of the first half of criterion (b), so we get that composition with g is an equivalence. This completes the proof of the claim.

We now return to the proof of the composition condition for (2). The fact that $\tau_{\leq 1}(A)$ is a 1-groupoid implies that given f there is another morphism h from y to x such that the class of fh is equal to the class of 1_y in $\pi_0 Hom_A(y, y)$, and the class of hf is equal to the class of 1_x in $\pi_0 Hom_A(x, x)$. This means that there exist 2-morphisms u from 1_y to fh , and v from 1_x to hf . By the above claim (and the fact that the compositions with 1_x and 1_y act as the identity and in particular are equivalences), we get that composition with fh is an equivalence

$$\{fh\} \times Hom_A(y, z) \rightarrow Hom_A(y, z),$$

and that composition with hf is an equivalence

$$\{hf\} \times Hom_A(x, z) \rightarrow Hom_A(x, z).$$

Let

$$\psi_f : Hom_A(y, z) \rightarrow Hom_A(x, z)$$

be the morphism of composition with f , and let

$$\psi_h : Hom_A(x, z) \rightarrow Hom_A(y, z)$$

be the morphism of composition with h . We have seen that $\psi_h\psi_f$ and $\psi_f\psi_h$ are equivalences. By the second sub-lemma (part (V) of the theorem) applied to $n - 1$ -groupoids, these imply that ψ_f is an equivalence.

The proof for composition in the other direction is the same; thus we have obtained condition (2) for A .

(2) \Rightarrow (1): Look at the condition (1) by referring to [20]: in question are the conditions $GR'_{i,k}$ and $GR''_{i,k}$ ($i < k \leq n$) of Definition 1.1, p. 33 of [20]. By the inductive version of the present equivalence for $n - 1$ -groupoids and by the part of condition (2) which says that the $Hom_A(x, y)$ are $n - 1$ -groupoids, we obtain the conditions $GR'_{i,k}$ and $GR''_{i,k}$ for $i \geq 1$. Thus we may now restrict our attention to the condition $GR'_{0,k}$ and $GR''_{0,k}$. For a 1-morphism a from x to y , the conditions $GR'_{0,k}$ for all k with respect to a , are the same as the condition that for all w , the morphism of pre-multiplication by a

$$Hom_A(w, x) \times \{a\} \rightarrow Hom_A(w, y)$$

is an equivalence according to the version (c) of the notion of equivalence (cf Part (III) of this theorem). Thus, condition $GR'_{0,k}$ follows from the second part of condition (2) (for pre-multiplication). Similarly condition $GR''_{0,k}$ follows from the second part of condition (2) for post-multiplication by every 1-morphism a . Thus condition (2) implies condition (1). This completes the proof of Part (I) of the theorem.

For the sub-lemma (part (IV) of the theorem), using the fact that isomorphisms of sets satisfy the same “three for two” property, and using the characterization of equivalences in terms of homotopy groups (condition (a)) we immediately get two of the three statements: that if f and g are equivalences then gf is an equivalence; and that if gf and g are equivalences then f is an equivalence. Suppose now that gf and f are equivalences; we would like to show that g is an equivalence. First of all it is clear that if $x \in Ob(A)$ then g induces an isomorphism $\pi_i(B, f(x)) \cong \pi_0(C, gf(x))$ (resp. $\pi_0(B) \cong \pi_0(C)$). Suppose now that $y \in Ob(B)$, and choose a 1-morphism u going from y to $f(x)$ for some $x \in Ob(A)$ (this is possible because f is surjective on π_0). By condition (2) for being a groupoid, composition with u induces equivalences along the top row of the diagram

$$\begin{array}{ccccc} Hom_B(y, y) & \rightarrow & Hom_B(y, f(x)) & \leftarrow & Hom_B(f(x), f(x)) \\ \downarrow & & \downarrow & & \downarrow \\ Hom_C(g(y), g(y)) & \rightarrow & Hom_C(g(y), gf(x)) & \leftarrow & Hom_C(gf(x), gf(x)). \end{array}$$

Similarly composition with $g(u)$ induces equivalences along the bottom row. The sub-lemma for $n - 1$ -groupoids applied to the sequence

$$Hom_A(x, x) \rightarrow Hom_B(f(x), f(x)) \rightarrow Hom_C(gf(x), gf(x))$$

as well as the hypothesis that f is an equivalence, imply that the rightmost vertical arrow in the above diagram is an equivalence. Again applying the sub-lemma to these $n - 1$ -groupoids yields that the leftmost vertical arrow is an equivalence. In particular g induces isomorphisms

$$\pi_i(B, y) = \pi_{i-1}(Hom_B(y, y), 1_y) \xrightarrow{\cong} \pi_{i-1}(Hom_C(g(y), g(y)), 1_{g(y)}) = \pi_i(C, g(y)).$$

This completes the verification of condition (a) for the morphism g , completing the proof of part (IV) of the theorem.

Finally we prove the second sub-lemma, part (V) of the theorem (from which we now adopt the notations A, B, C, D, f, g, h). Note first of all that applying π_0 gives the same situation for maps of sets, so $\pi_0(g)$ is an isomorphism. Next, suppose $x \in Ob(A)$. Then we obtain a sequence

$$\pi_i(A, x) \rightarrow \pi_i(B, f(x)) \rightarrow \pi_i(C, gf(x)) \rightarrow \pi_i(D, hgf(x)),$$

such that the composition of the first pair and also of the last pair are isomorphisms; thus g induces an isomorphism $\pi_i(B, f(x)) \cong \pi_i(C, gf(x))$. Now, by the same argument as for Part (IV) above, (using the hypothesis that f induces a surjection $\pi_0(A) \rightarrow \pi_0(B)$) we get that for any object $y \in Ob(B)$, g induces an isomorphism $\pi_i(B, y) \cong \pi_i(C, g(y))$. By definition (a) of Part (III) we have now shown that g is an equivalence. This completes the proof of the theorem. ///

Let $nStrGpd$ be the category of strict n -groupoids.

We close out this section by looking at how the groupoid condition fits in with the discussion of 1.2 and 1.3. Let C be a strict n -category with only one object x . Then C is an n -groupoid if and only if $Hom_C(x, x)$ is an $n - 1$ -groupoid and $\pi_0 Hom_C(x, x)$ (which has a structure of monoid) is a group. This is version (3) of the definition of groupoid in 2.1. Iterating this remark one more time we get the following statement.

Lemma 2.2 *The construction of 1.2 establishes an equivalence of categories between the strict n -groupoids having only one object and only one 1-morphism, and the abelian monoid-objects G in $(n - 2)StrGpd$ such that the monoid $\pi_0(G)$ is a group.* ///

Corollary 2.3 *Suppose C is a strict n -category having only one object and only one 1-morphism, and let B be the strict $n + 1$ -category of 1.3 with one object b and $Hom_B(b, b) = C$. Then B is a strict $n + 1$ -groupoid if and only if C is a strict n -groupoid.*

Proof: Keep the notations of the proof of 1.3. If C is a groupoid this means that G satisfies the condition that $\pi_0(G)$ be a group, which in turn implies that U is a groupoid. Note that $\pi_0(U) = *$ is automatically a group; so applying the observation 2.2 once again, we get that B is a groupoid. In the other direction, if B is a groupoid then $C = Hom_B(b, b)$ is a groupoid by versions (2) and (3) of the definition of groupoid. ///

3. Realization functors

Recall that $nStrGpd$ is the category of strict n -groupoids as defined above 2.1. Let Top be the category of topological spaces. The following definition encodes the minimum of what one would expect for a reasonable realization functor from strict n -groupoids to spaces.

Definition 3.1 *A realization functor for strict n -groupoids is a functor*

$$\mathfrak{R} : nStrGpd \rightarrow Top$$

together with the following natural transformations:

$$r : Ob(A) \rightarrow \mathfrak{R}(A);$$

$$\zeta_i(A, x) : \pi_i(A, x) \rightarrow \pi_i(\mathfrak{R}(A), r(x)),$$

the latter including $\zeta_0(A) : \pi_0(A) \rightarrow \pi_0(\mathfrak{R}(A))$; such that the $\zeta_i(A, x)$ and $\zeta_0(A)$ are isomorphisms for $0 \leq i \leq n$, and such that the $\pi_i(\mathfrak{R}(A), y)$ vanish for $i > n$.

Theorem 3.2 ([20]) *There exists a realization functor \mathfrak{R} for strict n -groupoids.*

Kapranov and Voevodsky [20] construct such a functor. Their construction proceeds by first defining a notion of “diagrammatic set”; they define a realization functor from n -groupoids to diagrammatic sets (denoted $Nerv$), and then define the topological realization of a diagrammatic set (denoted $|\cdot|$). The composition of these two constructions gives a realization functor

$$G \mapsto \mathfrak{R}_{KV}(G) := |Nerv(G)|$$

from strict n -groupoids to spaces. Note that this functor \mathfrak{R}_{KV} satisfies the axioms of 3.1 as a consequence of Propositions 2.7 and 3.5 of [20].

One obtains a different construction by considering strict n -groupoids as weak n -groupoids in the sense of [25] (multisimplicial sets) and then taking the realization of [25]. This construction is actually probably due to someone from the Australian school many years beforehand and we call it the *standard realization* \mathfrak{R}_{std} . The properties of 3.1 can be extracted from [25] (although again they are probably classical results).

We don’t claim here that any two realization functors must be the same, and in particular the realization \mathfrak{R}_{KV} could *a priori* be different from the standard one. This is why we shall work, in what follows, with an arbitrary realization functor satisfying the axioms of 3.1.

Here are some consequences of the axioms for a realization functor. If $C \rightarrow C'$ is a morphism of strict n -groupoids inducing isomorphisms on the π_i then $\mathfrak{R}(C) \rightarrow \mathfrak{R}(C')$ is a weak homotopy equivalence. Conversely if $f : C \rightarrow C'$ is a morphism of strict n -groupoids which induces a weak equivalence of realizations then f was an equivalence.

4. The case of the standard realization

Before getting to our main result which concerns an arbitrary realization functor satisfying 3.1, we take note of an easier argument which shows that the standard realization functor cannot give rise to arbitrary homotopy types.

Definition 4.1 A collection of realization functors \mathfrak{R}^n for n -groupoids ($0 \leq n < \infty$) satisfying 3.1 is said to be compatible with looping if there exist transformations natural in an n -groupoid A and an object $x \in \text{Ob}(A)$,

$$\varphi(A, x) : \mathfrak{R}^{n-1}(\text{Hom}_A(x, x)) \rightarrow \Omega^{r(x)} \mathfrak{R}^n(A)$$

(where $\Omega^{r(x)}$ means the space of loops based at $r(x)$), such that for $i \geq 1$ the following diagram commutes:

$$\begin{array}{ccc} \pi_i(A, x) & = \pi_{i-1}(\text{Hom}_A(x, x), 1_x) \rightarrow \pi_{i-1}(\mathfrak{R}^{n-1}(\text{Hom}_A(x, x)), r(1_x)) \\ \downarrow & & \downarrow \\ \pi_i(\mathfrak{R}^n(A), r(x)) & \leftarrow & \pi_{i-1}(\Omega^{r(x)} \mathfrak{R}^n(A), \text{cst}(r(x))) \end{array}$$

where the top arrow is $\zeta_{i-1}(\text{Hom}_A(x, x), 1_x)$, the left arrow is $\zeta_i(A, x)$, the right arrow is induced by $\varphi(A, x)$, and the bottom arrow is the canonical arrow from topology. (When $i = 1$, suppress the basepoints in the π_{i-1} in the diagram.)

Remark: The arrows on the top, the bottom and the left are isomorphisms in the above diagram, so the arrow on the right is an isomorphism and we obtain as a corollary of the definition that the $\varphi(A, x)$ are actually weak equivalences.

Remark: The collection of standard realizations $\mathfrak{R}_{\text{std}}^n$ for n -groupoids, is compatible with looping. We leave this as an exercise for the reader.

Recall the statements of 1.3 and 2.3: if A is a strict n -category with only one object x and only one 1-morphism 1_x , then there exists a strict $n + 1$ -category B with one object y , and with $\text{Hom}_B(y, y) = A$; and A is a strict n -groupoid if and only if B is a strict $n + 1$ -groupoid.

Corollary 4.2 Suppose $\{\mathfrak{R}^n\}$ is a collection of realization functors 3.1 compatible with looping 4.1. Then if A is a 1-connected strict n -groupoid (i.e. $\pi_0(A) = *$ and $\pi_1(A, x) = \{1\}$), the space $\mathfrak{R}^n(A)$ is weak-equivalent to a loop space.

Proof: Let $A' \subset A$ be the sub- n -category having one object x and one 1-morphism 1_x . For $i \geq 2$ the inclusion induces isomorphisms

$$\pi_i(A', x) \cong \pi_i(A, x),$$

and in view of the 1-connectedness of A this means (according to the definition of 2.1 III (a)) that the morphism $A' \rightarrow A$ is an equivalence. It follows (by definition 3.1) that $\mathfrak{R}^n(A') \rightarrow \mathfrak{R}^n(A)$ is a weak equivalence. Now A' satisfies the hypothesis of 1.3, 2.3 as recalled above, so there is an $n + 1$ -groupoid B having one object y such that $A' = \text{Hom}_B(y, y)$. By the definition of “compatible with looping” and the subsequent

remark that the morphism $\varphi(B, y)$ is a weak equivalence, we get that $\varphi(B, y)$ induces a weak equivalence

$$\mathfrak{R}^n(A') \rightarrow \Omega^{r(y)} \mathfrak{R}^{n+1}(B).$$

Thus $\mathfrak{R}^n(A)$ is weak-equivalent to the loop-space of $\mathfrak{R}^{n+1}(B)$. ///

The following corollary is a statement which seems to be due to C. Berger [8] (although the statement appears without proof in Grothendieck [18]). See also R. Brown and coauthors [9] [10] [11] [12].

Corollary 4.3 (C. Berger [8]) *There is no strict 3-groupoid A such that the standard realization $\mathfrak{R}_{\text{std}}(A)$ is weak-equivalent to the 3-type of S^2 .*

Proof: The 3-type of S^2 is not a loop-space. By the previous corollary (and the fact that the standard realizations are compatible with looping, which we have above left as an exercise for the reader), it is impossible for $\mathfrak{R}_{\text{std}}(A)$ to be the 3-type of S^2 . ///

5. Nonexistence of strict 3-groupoids giving rise to the 3-type of S^2

It is not completely clear whether Kapranov and Voevodsky claim that their realization functors are compatible with looping in the sense of 4.1, so Berger's negative result (Corollary 4.3 above) might not apply. The main work of the present paper is to extend this negative result to *any* realization functor satisfying the minimal definition 3.1, in particular getting a result which applies to the realization functor of [20].

Proposition 5.1 *Let \mathfrak{R} be any realization functor satisfying the properties of Definition 3.1. Then there does not exist a strict 3-groupoid C such that $\mathfrak{R}(C)$ is weak-equivalent to the 3-truncation of the homotopy type of S^2 .*

Corollary 5.2 *Let \mathfrak{R}_{KV} be the realization functor of Kapranov and Voevodsky [20] cf the discussion above. If we assume that Propositions 2.7 and 3.5 of [20] (stating that \mathfrak{R}_{KV} satisfies the axioms 3.1) are true, then Corollary 3.8 of [20] is not true, i.e. \mathfrak{R}_{KV} does not induce an equivalence between the homotopy categories of strict 3-groupoids and 3-truncated topological spaces.*

Proof: According to Proposition 5.1, for any realization functor satisfying 3.1, the induced functor on the homotopy categories is not essentially surjective: its essential image doesn't contain the 3-type of S^2 . ///

Proposition 5.1 is very similar to the result of Brown and Higgins [11] and also the recent result of C. Berger [8] (cf 4.3 above). As was noted in [20], the result of Brown

and Higgins concerns the more restrictive notion of groupoid where one requires that all morphisms have strict inverses (however, see also [9], [12]). As in [20], that restriction is not included in the definition 2.1. Berger considers strict n -groupoids according to the definition 2.1 (i.e. with inverses non-strict) as well, but his negative result applies only to a standard realization functor and as such, doesn't *a priori* directly contradict [20].

The basic difference in the present approach is that we make no reference to any particular construction of \mathfrak{R} but show that the proposition holds for any realization construction having the properties of Definition 3.1.

The fact that strict n -groupoids don't model all homotopy types is also mentioned in Grothendieck [18]. The basic idea in the setting of 3-categories not necessarily groupoids, is contained in some examples which G. Maltsiniotis pointed out to me, in Gordon-Power-Street [17] where there are given examples of weak 3-categories not equivalent to strict ones. This in turn is related to the difference between braided monoidal categories and symmetric monoidal categories, see for example the nice discussion in Baez-Dolan [2].

In order to prove 5.1, we will prove the following statement (which contains the main part of the argument). It basically says that the Postnikov tower of a simply connected strict 3-groupoid C , splits.

Proposition 5.3 *Suppose C is a strict 3-groupoid with an object c such that $\pi_0(C) = *$, $\pi_1(C, c) = \{1\}$, $\pi_2(C, c) = \mathbf{Z}$ and $\pi_3(C, c) = H$ for an abelian group H . Then there exists a diagram of strict 3-groupoids*

$$C \xleftarrow{g} B \xleftarrow{f} A \xrightarrow{h} D$$

*with objects $b \in \text{Ob}(B)$, $a \in \text{Ob}(A)$, $d \in \text{Ob}(D)$ such that $f(a) = b$, $g(b) = c$, $h(a) = d$. The diagram is such that g and f are equivalences of strict 3-groupoids, and such that $\pi_0(D) = *$, $\pi_1(D, d) = \{1\}$, $\pi_2(D, d) = \{0\}$, and such that h induces an isomorphism*

$$\pi_3(h) : \pi_3(A, a) = H \xrightarrow{\cong} \pi_3(D, d).$$

Proof of Proposition 5.1 using Proposition 5.3

Suppose for the moment that we know Proposition 5.3; with this we will prove 5.1. Fix a realization functor \mathfrak{R} for strict 3-groupoids satisfying the axioms 3.1, and assume that C is a strict 3-groupoid such that $\mathfrak{R}(C)$ is weak homotopy-equivalent to the 3-type of S^2 . We shall derive a contradiction.

In *résumé* the argument is this: that applying the realization functor to the diagram given by 5.3 and inverting the first two maps which are weak homotopy equivalences, we would get a map

$$\tau_{\leq 3}(S^2) = \mathfrak{R}(C) \rightarrow \mathfrak{R}(D) = K(H, 3)$$

(with $H = \mathbf{Z}$). This is a class in $H^3(S^2, H)$. The hypothesis that $\mathfrak{R}(h)$ is an isomorphism on π_3 means that this class is nonzero when applied to $\pi_3(S^2)$ via the Hurewicz homomorphism; but $H^3(S^2, \mathbf{Z}) = 0$, a contradiction.

Here is a full description of the argument. Apply Proposition 5.3 to C . Choose an object $c \in Ob(C)$. Note that, because of the isomorphisms between homotopy sets or groups 3.1, we have $\pi_0(C) = *$, $\pi_1(C, c) = \{1\}$, $\pi_2(C, c) = \mathbf{Z}$ and $\pi_3(C, c) = \mathbf{Z}$, so 5.3 applies with $H = \mathbf{Z}$. We obtain a sequence of strict 3-groupoids

$$C \xleftarrow{g} B \xleftarrow{f} A \xrightarrow{h} D.$$

This gives the diagram of spaces

$$\mathfrak{R}(C) \xleftarrow{\mathfrak{R}(g)} \mathfrak{R}(B) \xleftarrow{\mathfrak{R}(f)} \mathfrak{R}(A) \xrightarrow{\mathfrak{R}(h)} \mathfrak{R}(D).$$

The axioms 3.1 for \mathfrak{R} imply that \mathfrak{R} transforms equivalences of strict 3-groupoids into weak homotopy equivalences of spaces. Thus $\mathfrak{R}(f)$ and $\mathfrak{R}(g)$ are weak homotopy equivalences and we get that $\mathfrak{R}(A)$ is weak homotopy equivalent to the 3-type of S^2 .

On the other hand, again by the axioms 3.1, we have that $\mathfrak{R}(D)$ is 2-connected, and $\pi_3(\mathfrak{R}(D), r(d)) = H$ (via the isomorphism $\pi_3(D, d) \cong H$ induced by h, f and g). By the Hurewicz theorem there is a class $\eta \in H^3(\mathfrak{R}(D), H)$ which induces an isomorphism

$$\mathbf{Hur}(\eta) : \pi_3(\mathfrak{R}(D), r(d)) \xrightarrow{\cong} H.$$

Here

$$\mathbf{Hur} : H^3(X, H) \rightarrow Hom(\pi_3(X, x), H)$$

is the Hurewicz map for any pointed space (X, x) ; and the cohomology is singular cohomology (in particular it only depends on the weak homotopy type of the space).

Now look at the pullback of this class

$$\mathfrak{R}(h)^*(\eta) \in H^3(\mathfrak{R}(A), H).$$

The hypothesis that $\mathfrak{R}(u)$ induces an isomorphism on π_3 implies that

$$\mathbf{Hur}(\mathfrak{R}(h)^*(\eta)) : \pi_3(\mathfrak{R}(A), r(a)) \xrightarrow{\cong} H.$$

In particular, $\mathbf{Hur}(\mathfrak{R}(h)^*(\eta))$ is nonzero so $\mathfrak{R}(h)^*(\eta)$ is nonzero in $H^3(\mathfrak{R}(A), H)$. This is a contradiction because $\mathfrak{R}(A)$ is weak homotopy-equivalent to the 3-type of S^2 , and $H = \mathbf{Z}$, but $H^3(S^2, \mathbf{Z}) = \{0\}$.

This contradiction completes the proof of Proposition 5.1, assuming Proposition 5.3.

///

Proof of Proposition 5.3

This is the main part of the argument. We start with a strict groupoid C and object c , satisfying the hypotheses of 5.3.

The first step is to construct (B, b) . We let $B \subset C$ be the sub-3-category having only one object $b = c$, and only one 1-morphism $1_b = 1_c$. We set

$$Hom_{Hom_B(b,b)}(1_b, 1_b) := Hom_{Hom_C(c,c)}(1_c, 1_c),$$

with the same composition law. The map $g : B \rightarrow C$ is the inclusion.

Note first of all that B is a strict 3-groupoid. This is easily seen using version (1) of the definition 2.1 (but one has to look at the conditions in [20]). We can also verify it using condition (3). Of course $\tau_{\leq 1}(B)$ is the 1-category with only one object and only one morphism, so it is a groupoid. We have to verify that $Hom_B(b, b)$ is a strict 2-groupoid. For this, we again apply condition (3) of 2.1. Here we note that

$$Hom_B(b, b) \subset Hom_C(c, c)$$

is the full sub-2-category with only one object $1_b = 1_c$. Therefore, in view of the definition of $\tau_{\leq 1}$, we have that

$$\tau_{\leq 1}Hom_B(b, b) \subset \tau_{\leq 1}Hom_C(c, c)$$

is a full subcategory. A full subcategory of a 1-groupoid is again a 1-groupoid, so $\tau_{\leq 1}Hom_B(b, b)$ is a 1-groupoid. Finally, $Hom_{Hom_B(b,b)}(1_b, 1_b)$ is a 1-groupoid since by construction it is the same as $Hom_{Hom_C(c,c)}(1_c, 1_c)$ (which is a groupoid by condition (3) applied to the strict 2-groupoid $Hom_C(c, c)$). This shows that $Hom_B(b, b)$ is a strict 2-groupoid and hence that B is a strict 3-groupoid.

Next, note that $\pi_0(B) = *$ and $\pi_1(B, b) = \{1\}$. On the other hand, for $i = 2, 3$ we have

$$\pi_i(B, b) = \pi_{i-2}(Hom_{Hom_B(b,b)}(1_b, 1_b), 1_b^2)$$

and similarly

$$\pi_i(C, c) = \pi_{i-2}(Hom_{Hom_C(c,c)}(1_c, 1_c), 1_c^2),$$

so the inclusion g induces an equality $\pi_i(B, b) \xrightarrow{=} \pi_i(C, c)$. Therefore, by definition (a) of equivalence 2.1, g is an equivalence of strict 3-groupoids. This completes the construction and verification for B and g .

Before getting to the construction of A and f , we analyze the strict 3-groupoid B in terms of the discussion of 1.2 and 2.2. Let

$$G := Hom_{Hom_B(b,b)}(1_b, 1_b).$$

It is an abelian monoid-object in the category of 1-groupoids, with abelian operation denoted by $+$: $G \times G \rightarrow G$ and unit element denoted $0 \in G$ which is the same as 1_b . The operation $+$ corresponds to both of the compositions $*_0$ and $*_1$ in B .

The hypotheses on the homotopy groups of C also hold for B (since g was an equivalence). These translate to the statements that $(\pi_0(G), +) = \mathbf{Z}$ and $\text{Hom}_G(0, 0) = H$.

We now construct A and f via 1.2 and 2.2, by constructing a morphism $(G', +) \rightarrow (G, +)$ of abelian monoid-objects in the category of 1-groupoids. We do this by a type of “base-change” on the monoid of objects, i.e. we will first define a morphism $\text{Ob}(G') \rightarrow \text{Ob}(G)$ and then define G' to be the groupoid with object set $\text{Ob}(G')$ but with morphisms corresponding to those of G .

To accomplish the “base-change”, start with the following construction. If S is a set, let $\mathbf{E}(S)$ denote the groupoid with S as set of objects, and with exactly one morphism between each pair of objects. If S has an abelian monoid structure then $\mathbf{E}(S)$ is an abelian monoid object in the category of groupoids.

Note that for any groupoid U there is a morphism of groupoids

$$U \rightarrow \mathbf{E}(\text{Ob}(U)),$$

and by “base change” we mean the following operation: take a set S with a map $p : S \rightarrow \text{Ob}(U)$ and look at

$$V := \mathbf{E}(S) \times_{\mathbf{E}(\text{Ob}(U))} U.$$

This is a groupoid with S as set of objects, and with

$$\text{Hom}_V(s, t) = \text{Hom}_U(p(s), p(t)).$$

If U is an abelian monoid object in the category of groupoids, if S is an abelian monoid and if p is a map of monoids then V is again an abelian monoid object in the category of groupoids.

Apply this as follows. Starting with $(G, +)$ corresponding to B via 1.2 and 2.2 as above, choose objects $a, b \in \text{Ob}(G)$ such that the image of a in $\pi_0(G) \cong \mathbf{Z}$ corresponds to $1 \in \mathbf{Z}$, and such that the image of b in $\pi_0(G)$ corresponds to $-1 \in \mathbf{Z}$. Let N denote the abelian monoid, product of two copies of the natural numbers, with objects denoted (m, n) for nonnegative integers m, n . Define a map of abelian monoids

$$p : N \rightarrow \text{Ob}(G)$$

by

$$p(m, n) := m \cdot a + n \cdot b := a + a + \dots + a + b + b + \dots + b.$$

Note that this induces the surjection $N \rightarrow \pi_0(G) = \mathbf{Z}$ given by $(m, n) \mapsto m - n$.

Define $(G', +)$ as the base-change

$$G' := \mathbf{E}(N) \times_{\mathbf{E}(Ob(G))} G,$$

with its induced abelian monoid operation $+$. We have

$$Ob(G') = N,$$

and the second projection $p_2 : G' \rightarrow G$ (which induces p on object sets) is fully faithful i.e.

$$Hom_{G'}((m, n), (m', n')) = Hom_G(p(m, n), p(m', n')).$$

Note that $\pi_0(G') = \mathbf{Z}$ via the map induced by p or equivalently p_2 . To prove this, say that: (i) N surjects onto \mathbf{Z} so the map induced by p is surjective; and (ii) the fact that p_2 is fully faithful implies that the induced map $\pi_0(G') \rightarrow \pi_0(G) = \mathbf{Z}$ is injective.

We let A be the strict 3-groupoid corresponding to $(G', +)$ via 1.2, and let $f : A \rightarrow B$ be the map corresponding to $p_2 : G' \rightarrow G$ again via 1.2. Let a be the unique object of A (it is mapped by f to the unique object $b \in Ob(B)$).

The fact that $(\pi_0(G'), +) = \mathbf{Z}$ is a group implies that A is a strict 3-groupoid (2.2). We have $\pi_0(A) = *$ and $\pi_1(A, a) = \{1\}$. Also,

$$\pi_2(A, a) = (\pi_0(G'), +) = \mathbf{Z}$$

and f induces an isomorphism from here to $\pi_2(B, b) = (\pi_0(G), +) = \mathbf{Z}$. Finally (using the notation $(0, 0)$ for the unit object of $(N, +)$ and the notation 0 for the unit object of $Ob(G)$),

$$\pi_3(A, a) = Hom_{G'}((0, 0), (0, 0)),$$

and similarly

$$\pi_3(B, b) = Hom_G(0, 0) = H;$$

the map $\pi_3(f) : \pi_3(A, a) \rightarrow \pi_3(B, b)$ is an isomorphism because it is the same as the map

$$Hom_{G'}((0, 0), (0, 0)) \rightarrow Hom_G(0, 0)$$

induced by $p_2 : G' \rightarrow G$, and p_2 is fully faithful. We have now completed the verification that f induces isomorphisms on the homotopy groups, so by version (a) of the definition of equivalence 2.1, f is an equivalence of strict 3-groupoids.

We now construct D and define the map h by an explicit calculation in $(G', +)$. First of all, let $[H]$ denote the 1-groupoid with one object denoted 0 , and with H as group of endomorphisms:

$$Hom_{[H]}(0, 0) := H.$$

This has a structure of abelian monoid-object in the category of groupoids, denoted $([H], +)$, because H is an abelian group. Let D be the strict 3-groupoid corresponding to $([H], +)$ via 1.2 and 2.2. We will construct a morphism $h : A \rightarrow D$ via 1.2 by constructing a morphism of abelian monoid objects in the category of groupoids,

$$h : (G', +) \rightarrow ([H], +).$$

We will construct this morphism so that it induces the identity morphism

$$Hom_{G'}((0, 0), (0, 0)) = H \rightarrow Hom_{[H]}(0, 0) = H.$$

This will insure that the morphism h has the property required for 5.3.

The object $(1, 1) \in N$ goes to $0 \in \pi_0(G') \cong \mathbf{Z}$. Thus we may choose an isomorphism $\varphi : (0, 0) \cong (1, 1)$ in G' . For any k let $k\varphi$ denote the isomorphism $\varphi + \dots + \varphi$ (k times) going from $(0, 0)$ to (k, k) . On the other hand, H is the automorphism group of $(0, 0)$ in G' . The operations $+$ and composition coincide on H . Finally, for any $(m, n) \in N$ let $1_{m,n}$ denote the identity automorphism of the object (m, n) . Then any arrow α in G may be uniquely written in the form

$$\alpha = 1_{m,n} + k\varphi + u$$

with (m, n) the source of α , the target being $(m + k, n + k)$, and where $u \in H$.

We have the following formulae for the composition \circ of arrows in G' . They all come from the basic rule

$$(\alpha \circ \beta) + (\alpha' \circ \beta') = (\alpha + \alpha') \circ (\beta + \beta')$$

which in turn comes simply from the fact that $+$ is a morphism of groupoids $G' \times G' \rightarrow G'$ defined on the cartesian product of two copies of G . Note in a similar vein that $1_{0,0}$ acts as the identity for the operation $+$ on arrows, and also that

$$1_{m,n} + 1_{m',n'} = 1_{m+m',n+n'}.$$

Our first equation is

$$(1_{l,l} + k\varphi) \circ l\varphi = (k + l)\varphi.$$

To prove this note that $l\varphi + 1_{0,0} = l\varphi$ and our basic formula says

$$(1_{l,l} \circ l\varphi) + (k\varphi \circ 1_{0,0}) = (1_{l,l} + k\varphi) \circ (l\varphi + 1_{0,0})$$

but the left side is just $l\varphi + k\varphi = (k + l)\varphi$.

Now our basic formula, for a composition starting with (m, n) , going first to $(m + l, n + l)$, then going to $(m + l + k, n + l + k)$, gives

$$(1_{m+l,n+l} + k\varphi + u) \circ (1_{m,n} + l\varphi + v)$$

$$\begin{aligned}
&= (1_{m,n} + 1_{l,l} + k\varphi + u) \circ (1_{m,n} + l\varphi + v) \\
&= 1_{m,n} \circ 1_{m,n} + (1_{l,l} + k\varphi) \circ l\varphi + u \circ v \\
&= 1_{m,n} + (k + l)\varphi + (u \circ v)
\end{aligned}$$

where of course $u \circ v = u + v$.

This formula shows that the morphism h from arrows of G' to the group H , defined by

$$h(1_{m,n} + k\varphi + u) := u$$

is compatible with composition. This implies that it provides a morphism of groupoids $h : G \rightarrow [H]$ (recall from above that $[H]$ is defined to be the groupoid with one object whose automorphism group is H). Furthermore the morphism h is obviously compatible with the operation $+$ since

$$\begin{aligned}
&(1_{m,n} + k\varphi + u) + (1_{m',n'} + k'\varphi + u') = \\
&\quad (1_{m+m',n+n'} + (k + k')\varphi + (u + u'))
\end{aligned}$$

and once again $u + u' = u \circ u'$ (the operation $+$ on $[H]$ being given by the commutative operation \circ on H).

This completes the construction of a morphism $h : (G, +) \rightarrow ([H], +)$ which induces the identity on $Hom(0, 0)$. This corresponds to a morphism of strict 3-groupoids $h : A \rightarrow D$ as required to complete the proof of Proposition 5.3. ///

6. A remark on strict ∞ -groupoids

The nonexistence result of 5.1 holds also for strict ∞ -groupoids as defined in [20]. Recall that Kapranov-Voevodsky [20] extend the notion of strict n -category and strict n -groupoid to the case $n = \infty$. The definition is made using condition (1), and the notion of equivalence is defined using (a) in 2.1. Note that the other characterizations of 2.1 don't actually make sense in the case $n = \infty$ because they are inductive on n .

The only thing we need to know about the case $n = \infty$ is that there are homotopy groups $\pi_i(A, a)$ of a strict ∞ -groupoid A , and there are truncation operations on strict ∞ -groupoids such that $\tau_{\leq n}(A)$ is a strict n -groupoid with a natural morphism

$$A \rightarrow \tau_{\leq n}(A)$$

inducing isomorphisms on homotopy groups for $i \leq n$. (Here the n -groupoid $\tau_{\leq n}(A)$ is considered as an ∞ -groupoid in the obvious way.) The homotopy groups and truncation are defined as in [20]—again, one has to avoid those versions of the definitions 2.1 which are recursive on n .

We can extend the definition of 3.1 to the case $n = \infty$. It is immediate that for any realization functor \mathfrak{R} satisfying the axioms 3.1 for $n = \infty$, the morphism

$$\mathfrak{R}(A) \rightarrow \mathfrak{R}(\tau_{\leq n}A)$$

is the Postnikov truncation of $\mathfrak{R}(A)$. Applying 5.1, we obtain the following result.

Corollary 6.1 *For any realization functor \mathfrak{R} satisfying the axioms 3.1 for $n = \infty$, there does not exist a strict ∞ -groupoid A (as defined by Kapranov-Voevodsky [20]) such that $\mathfrak{R}(A)$ is weak homotopy-equivalent to the 2-sphere S^2 .*

Proof: Note that if \mathfrak{R} is a realization functor satisfying 3.1 for $n = \infty$, then composing with the inclusion i_3^∞ from the category of strict 3-groupoids to the category of strict ∞ -groupoids we obtain a realization functor $\mathfrak{R}i_3^\infty$ for strict 3-groupoids, again satisfying 3.1. If A is a strict ∞ -groupoid then the above truncation morphism, written more precisely, is

$$A \rightarrow i_3^\infty \tau_{\leq 3}(A).$$

This induces isomorphisms on the π_i for $i \leq 3$. Applying \mathfrak{R} we get

$$\mathfrak{R}(A) \rightarrow \mathfrak{R}i_3^\infty \tau_{\leq 3}(A),$$

inducing an isomorphism on homotopy groups for $i \leq 3$. In particular, if $\mathfrak{R}(A)$ were weak homotopy-equivalent to S^2 then this would imply that $\mathfrak{R}i_3^\infty \tau_{\leq 3}(A)$ is the 3-type of S^2 . In

view of the fact that \mathfrak{R}_3^∞ is a realization functor according to 3.1 for strict 3-groupoids, this would contradict 5.1. Thus we conclude that there is no strict ∞ -groupoid A with $\mathfrak{R}(A)$ weak homotopy-equivalent to S^2 . ///

7. Conclusion

One really needs to look at some type of weak 3-categories in order to get a hold of 3-truncated homotopy types. O. Leroy [22] and apparently, independantly, Joyal and Tierney [19] were the first to do this. See also Gordon, Power, Street [17] and Berger [8] for weak 3-categories and 3-types. Baues [6] showed that 3-types correspond to *quadratic modules* (a generalization of the notion of crossed complex) [6]. Tamsamani [25] was the first to relate weak n -groupoids and homotopy n -types. For other notions of weak n -category, see [1] [3] [4], [5].

From homotopy theory (cf [23]) the following type of yoga seems to come out: that it suffices to weaken any one of the principal structures involved. Most weak notions of n -category involve a weakening of the associativity, or eventually of the Godement (commutativity) conditions.

It seems likely that the arguments of [20] would show that one could instead weaken the condition of being *unary* (i.e. having identities for the operations) and keep associativity and Godement. We give a proposed definition of what this would mean and then state two conjectures.

Motivation

Before giving the definition, we motivate these remarks by looking at the *Moore loop space* $\Omega_M^x(X)$ of a space X based at $x \in X$ (the Moore loop space is referred to in [20] as a motivation for their construction). Recall that $\Omega_M^x(X)$ is the space of *pairs* (r, γ) where r is a real number $r \geq 0$ and $\gamma = [0, r] \rightarrow X$ is a path starting and ending at x . This has the advantage of being a strictly associative monoid. On the other side of the coin, the “length” function

$$\ell : \Omega_M^x(X) \rightarrow [0, \infty) \subset \mathbf{R}$$

has a special behavoir over $r = 0$. Note that over the open half-line $(0, \infty)$ the length function ℓ is a fibration (even a fiber-space) with fiber homeomorphic to the usual loop space. However, the fiber over $r = 0$ consists of a single point, the constant path $[0, 0] \rightarrow X$ based at x . This additional point (which is the unit element of the monoid $\Omega_M^x(X)$) doesn’t affect the topology of $\Omega_M^x(X)$ (at least if X is locally contractible at x) because it is glued in as a limit of paths which are more and more concentrated in a neighborhood of x . However, the map ℓ is no longer a fibration over a neighborhood of $r = 0$. This is a bit of

a problem because Ω_M^x is not compatible with direct products of the space X ; in order to obtain a compatibility one has to take the fiber product over \mathbf{R} via the length function:

$$\Omega_M^{(x,y)}(X \times Y) = \Omega_M^x(X) \times_{\mathbf{R}} \Omega_M^y(Y),$$

and the fact that ℓ is not a fibration could end up causing a problem in an attempt to iteratively apply a construction like the Moore loop-space.

Things seem to get better if we restrict to

$$\Omega_{M'}^x(X) := \ell^{-1}((0, \infty)) \subset \Omega_M^x(X),$$

but this associative monoid no longer has a strict unit. Even so, the constant path of any positive length gives a weak unit.

A motivation coming from a different direction was an observation made by Z. Tamsamani early in the course of doing his thesis. He was trying to define a strict 3-category $2Cat$ whose objects would be the strict 2-categories and whose morphisms would be the weak 2-functors between 2-categories (plus notions of weak natural transformations and 2-natural transformations). At some point he came to the conclusion that one could adequately define $2Cat$ as a strict 3-category except that he couldn't get strict identities. Because of this problem we abandoned the idea and looked toward weakly associative n -categories. In retrospect it would be interesting to pursue Tamsamani's construction of a strict $2Cat$ but with only weak identities.

Snucategories

Now we get back to looking at what it could mean to weaken the unit property for strict n -categories or strict n -groupoids. We will define a notion of n -snucategory (the initial 's' stands for strict, 'nu' stands for non-unary) by induction on n . There will be a notion of direct product of n -snucategories. Suppose we know what these mean for $n - 1$. Then an n -snucategory C consists of a set C_0 of objects together with, for every pair of objects $x, y \in C_0$ an $n - 1$ -snucategory $Hom_C(x, y)$ and composition morphisms

$$Hom_C(x, y) \times Hom_C(y, z) \rightarrow Hom_C(x, z)$$

which are strictly associative, such that the *weak unary condition* is satisfied. We now explain this condition. An element $e_x \in Hom_C(x, x)$ is called a weak identity if:

—composition with e induces equivalences of $n - 1$ -snucategories

$$Hom_C(x, y) \rightarrow Hom_C(x, y), \quad Hom_C(y, x) \rightarrow Hom_C(y, x);$$

—and if $e \cdot e$ is equivalent to e .

In order to complete the recursive definition we must define the notion of when a morphism of n -snucategories is an equivalence, and we must define what it means for two objects to be equivalent. A morphism is said to be an equivalence if the induced morphisms on Hom are equivalences of $n-1$ -snucategories and if it is essentially surjective on objects: each object in the target is equivalent to the image of an object. It thus remains just to be seen what equivalence of objects means. For this we introduce the *truncations* $\tau_{\leq i}C$ of an n -snucategory C . Again this is done in the same way as usual: $\tau_{\leq i}C$ is the i -snucategory with the same objects as C and whose Hom 's are the truncations

$$Hom_{\tau_{\leq i}C}(x, y) := \tau_{\leq i-1}Hom_C(x, y).$$

This works for $i \geq 1$ by recurrence, and for $i = 0$ we define the truncation to be the set of isomorphism classes in $\tau_{\leq 1}C$. Note that truncation is compatible with direct product (direct products are defined in the obvious way) and takes equivalences to equivalences. These statements used recursively allow us to show that the truncations themselves satisfy the weak unary condition. Finally, we say that two objects are equivalent if they map to the same thing in $\tau_{\leq 0}C$.

Proceeding in the same way as in §2 above, we can define the notion of n -snugroupoid.

Conjecture 1 *There are functors Π_n and \mathfrak{R} between the categories of n -snugroupoids and n -truncated spaces (going in the usual directions) together with adjunction morphisms inducing an equivalence between the localization of n -snugroupoids by equivalences, and n -truncated spaces by weak equivalences.*

I think that the argument of [20] (which is unclear on the question of identity elements) actually serves to prove the above statement. I have called the above statement a “conjecture” because I haven’t checked this.

One might go out on a limb a bit more and make the following

Conjecture 2 *The localization of the category of n -snucategories by equivalences is equivalent to the localizations of the categories of weak n -categories of Tamsamani and/or Baez-Dolan and/or Batanin by equivalences.*

This of course is of a considerably more speculative nature.

Caveat: the above definition of “snucategory” is invented in an *ad hoc* way, and in particular one naturally wonders whether or not the equivalences $e \cdot e \sim e$ and higher homotopical data going along with that, would need to be specified in order to get a good definition. I have no opinion about this (the above definition being just the easiest thing to say which gives some idea of what needs to be done). Thus it is not completely clear that the above definition of n -snucategory is the “right” one to fit into the conjectures.

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